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Find general sols. to the following DE's :

1. $y'' + 3y' = e^{\frac{x}{2}}$

2. $y'' + 9y = 9x^4 - 9$

3. $y'' - 4y' + 5y = 10x^2 + 4x + 8$

Sol: 1. ① Guess one particular sol. y_p.

Observe RHS is of the form $e^{\frac{x}{2}}$.

Consider $y_p = c\,e^{\frac{x}{2}}$ $c \in \mathbb{R}$.

$$y_p'' + 3y_p' = \left(\frac{c}{4} + \frac{3c}{2}\right)e^{\frac{x}{2}} = e^{\frac{x}{2}}$$

$$\Rightarrow \quad c = \frac{4}{7}$$

So $y_p = \frac{4}{7} e^{\frac{x}{2}}$ is a particular sol.

② Consider $y'' + 3y' = 0$ (H).

$\boxed{\text{Method 1 : Characteristic polynomial}}$: $\lambda^2 + 3\lambda = 0$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \lambda = -3$$

So $y_0 = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot e^{\lambda_2 x} = c_1 + c_2 e^{-3x}$ is general sol. for (H).

$\boxed{\text{Method 2 : Integrating factor :}}$

Let $g = y'$. then $g' + 3g = 0$

$\boxed{\text{Idea :}}$ multiply by $\mu(x)$ so that $g'\mu + 3\mu g = (\mu g)'$

According to this, $\mu'(x) = 3\mu(x)$

$$\Rightarrow \frac{\mu'(x)}{\mu(x)} = 3$$

$$\Rightarrow \mu(x) = c \cdot e^{3x}. \quad c \in \mathbb{R}.$$

Take $c = 1$

$$\Rightarrow (e^{3x} \cdot g)' = \mu g' + 3\mu g = 0 \cdot \mu = 0$$

i.e. $(e^{3x} \cdot g)' = 0$

$$\Rightarrow g \cdot e^{3x} = c_1, \quad c_1 \text{ is a constant}$$

$$\Rightarrow g(x) = c_1 e^{-3x}$$

Now $y'(x) = g(x) = c_1 e^{-3x}$

$$\Rightarrow y(x) = \int c_1 e^{-3x} + c_2$$

$$= \tilde{c}_1 e^{-3x} + c_2, \quad \tilde{c}_1, c_2 \in \mathbb{R}.$$

③ General sol. for original ODE :

$$y = y_0 + y_p$$
$$= C_1 e^{-3x} + C_2 + \frac{4}{7} e^{\frac{x}{2}}, \quad C_1, C_2 \in \mathbb{R}.$$

Remark: If y_1, y_2 are sol's to the ODE, $\underset{\text{homogeneous}}{(H)}$ and $\underset{\text{nonhomogeneous}}{(NH)}$ respectively, then $y_1 + y_2$ is ~~also~~ a solution to (NH).

$$(y_1 + y_2)'' + 3(y_1 + y_2)' = y_1'' + y_2'' + 3y_1' + 3y_2'$$
$$= (y_1'' + 3y_1') + (y_2'' + 3y_2').$$

Note: $y_1'' + 3y_1' = 0$

$$y_2'' + 3y_2' = e^{\frac{x}{2}}$$
$$\Rightarrow (y_1 + y_2)'' = e^{\frac{x}{2}}.$$

Sol 2: ① characteristic polynomial : $\lambda^2 + 9 = 0$

$$\Rightarrow \lambda = \pm 3i, \quad \text{where } i^2 = -1$$

So e^{3ix} and e^{-3ix} are sols to (H) : $y'' + 9y = 0$.

> Note: $e^{3ix} = \cos(3x) + i\sin(3x)$
>
> $f = u + iv$ is a sol. to (H)
>
> $\Rightarrow u, v$ are sols to (H).

Reason: $f'' = u'' + iv''$

$$f'' + 9f = (u'' + iv'') + 9(u + iv)$$
$$= (u'' + 9u) + i(v'' + 9v) = 0$$

So $\begin{cases} u'' + 9u = 0 \\ v'' + 9v = 0 \end{cases}$

$\Rightarrow u, v$ are sols to (H).

Also, $\cos 3x$ and $\sin 3x$ are linearly independent. (2 vectors not parallel to each other)

so

$$y = C_1 \cos(3x) + C_2 \cdot \sin(3x), \quad C_1, C_2 \in \mathbb{R}$$

② Guess one particular sol. y_p.

Observe RHS is a polynomial of order 4.

Try : $y = ax^4 + bx^3 + cx^2 + dx + e$.

$y'' = 12ax^2 + 6bx + 2c$

$y'' + 9y = (12ax^2 + 6bx + 2c) + 9(ax^4 + bx^3 + cx^2 + dx + e) = 9x^4 - 9$

Compare coefficients :

$$\begin{cases} \bullet\ 9a = 9 \\ \bullet\ 9b = 0 \\ 12a + 9c = 0 \\ 6b + 9d = 0 \\ 2c + 9e = -9 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ c = \cancel{-\frac{3}{4}}\ -\frac{4}{3} \\ d = 0 \\ e = \cancel{-\frac{5}{6}}\ -\frac{19}{27} \end{cases}$$

③ $y = y_0 + y_p$

$= C_1 \cos 3x + C_2 \sin 3x + \left(x^4 - \frac{3}{4}x^2 - \frac{5}{6} \right)$, $C_1, C_2 \in \mathbb{R}$.

Sol $\cancel{2}$': $y'' - 9y = 9x^4 - 9$

① char. poly. : $\lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$

So $y_0 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = C_1 e^{3x} + C_2 e^{-3x}$, $C_1, C_2 \in \mathbb{R}$.

② Guess $y_p = ax^4 + bx^3 + cx^2 + dx + e$.

$y_p'' - 9y_p = 9x^4 - 9 \Rightarrow \begin{cases} a = -1 , & d = 0 \\ b = 0 , & e = \cancel{\frac{5}{6}}\ \frac{19}{27} \\ c = \cancel{\frac{3}{4}}\ -\frac{4}{3} & \end{cases}$

③ $y = y_0 + y_p$

Sol 3. $y'' - 4y' + 5y = 10x^2 + 4x + 8$

① char. poly.: $\lambda^2 - 4\lambda + 5 = 0$

$\Rightarrow \quad \lambda = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\lambda_1 = 2 + i, \qquad \lambda_2 = 2 - i$

Note: $e^{\lambda_1 x} = e^{(2+i)x} = e^{2x} \cdot e^{ix} = e^{2x}(\cos x + i \sin x)$

The real part and imaginary part are both sol's to (H).

Hence $y_0 = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x, \quad C_1, C_2 \in \mathbb{R}$.

$= e^{2x}(C_1 \cos x + C_2 \sin x)$.

[Verify this by yourself !]

② Guess $y_p = ax^2 + bx + c$

$y_p'' = 2a, \quad y_p' = 2ax + b, \quad y_p = ax^2 + bx + c$.

$\Rightarrow \quad y_p'' - 4y_p' + 5y_p = (2a - 8ax - 4b + 5ax^2 + 5bx + 5c) = 10x^2 + 4x + 8$

So $\begin{cases} 5a = 10 \\ -8a + 5b = 4 \\ 2a - 4b + 5c = 8 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = 4 \\ c = 4 \end{cases}$

③ $y = y_0 + y_p = e^{2x}(C_1 \cos x + C_2 \sin x) + (2x^2 + 4x + 4), \quad C_1, C_2 \in \mathbb{R}$.

Sol 3': $y'' - 4y' - 5y = 10x^2 + 4x + 8$

① char. poly. $\lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \lambda_1 = 5 \qquad \lambda_2 = -1$

$\Rightarrow \quad y_0 = C_1 e^{5x} + C_2 e^{-x}, \qquad C_1, C_2 \in \mathbb{R}$

② Try $y_p = ax^2 + bx + c$

$\Rightarrow \begin{cases} -5a = 10 \\ -8a - 5b = 4 \\ 2a - 4b - 5c = 8 \end{cases} \Rightarrow \begin{cases} a = -2 \\ b = \frac{12}{5} \\ c = -\frac{108}{25} \end{cases}$

③ $y = y_0 + y_p$.